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Manuscript received June 8, 1972; paper accepted July 20, 1972.

Suboptimal Feedback Control of Distributed Systems:

Part I. Theoretical Developments

A new method for suboptimal feedback control of certain classes of distributed systems based on successive instantaneous minimization of the performance functional kernel is presented. This method is applicable to both linear and nonlinear systems.

The Lyapunov functional approach is extended to distributed systems, yielding two new suboptimal feedback control techniques. Also, an application of multilevel bang-bang control to the first-order hyperbolic system is presented and generalized.

Advantages and disadvantages of the suboptimal control techniques vis á vis open loop optimal control methods are listed, as well as comparative features of the suboptimal control methods.

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SCOPE

Many processes encountered in engineering practice possess significantly nonuniform spatial distributions in the values of their state variables. In modeling the unsteady state behavior of such systems, the use of partial differential equations is very desirable. Although ensuring an accurate representation of system behavior, the use of a distributed model makes the synthesis of an optimal control policy for the system extremely laborious. Furthermore, except in one special case, the optimal control must be applied to the system in a feedforward manner. Feedforward control is plagued by many shortcomings, rendering it unsatisfactory for many practical applications.

The objective of this work is to develop feedback control algorithms for distributed systems which provide near-optimal control, yet which are conceptually simple and flexible enough to have potential for practical use.

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Extensions of existing techniques for lumped systems are made in the development of suboptimal control algorithms utilizing the Lyapunov functional, and original techniques are employed in the development of algorithms involving instantaneous minimization of the performance functional kernel.

For all cases, suboptimal control is sought by relaxation of optimality criteria, rather than by approximation to a rigorously optimal control, as in many previous works.

CONCLUSIONS AND SIGNIFICANCE

Three basic types of suboptimal feedback control algorithms applicable to distributed systems are presented.

1. Techniques based on the minimization of the time derivative of a Lyapunov functional. These are extensions of existing techniques for lumped systems.
2. Multilevel bang-bang technique. This may be considered as a special case of No. 1.
3. Instantaneous kernel minimization techniques. Only one particular technique of a whole class is developed in this work. These techniques, formulated for use with parabolic systems, depend on the utilization of one or more criteria for optimality which are obtained by relaxation of the classical criteria in order to provide a form of the control which may be evaluated from the state of the system measured at any instant in time.

In principle, all of these techniques are capable of providing near optimal control for either linear or nonlinear distributed systems in feedback form. Feedback operation provides many advantages which enhance the practical utility of these techniques. Among these advantages are:

1. Small parameter variations in the controlled system will not drastically affect the quality of control.
2. A system model which is not an exact representation of the actual system will not yield a control which is totally useless.
3. Only modest computational capability is required. There is no demand whatever for large information storage capability, as in feedforward optimal control.
4. The free parameters included in these control schemes allow their adjustment to suit any particular controlled system.

Due to the simplicity and versatility of the algorithms, they may be considered suitable for practical applications in which distributed systems are to be controlled.

MOTIVATION

At the present time, all techniques for the synthesis of optimal controls in distributed parameter systems, save the Riccati transformation approach to the linear-quadratic problem, produce the optimal control in open-loop form. The entire optimal control as a function of time is precomputed and stored off-line, to be applied to the system at hand at a later time. Due to the disadvantages inherent in the use of open-loop control and to the complexity and limited applicability of the Riccati transformation technique, the development of algorithms for optimal or near-optimal feedback control of a wide range of distributed systems is of particular interest.

This paper presents three such algorithms, each of which operates on the state of a distributed system to yield a feedback control. In each case, the advantage of closed-loop operation is gained at the expense of a slight relaxation of the classical criteria for optimality.

To underscore the potential utility of suboptimal feedback control for distributed systems, we first discuss the disadvantages of open loop control.

1. The off-line optimal control is synthesized using a mathematical model which is an idealization of the system under study. If the actual system is not precisely known, the precomputed control will not be optimal.
2. A new precomputed optimal control is required if the initial and/or boundary conditions of the system do not match original estimates.
3. Stochastic disturbances and parameter variations cannot be compensated for with existing methods.
4. Synthesis of off-line optimal controls for distributed systems requires considerable computational capability.
5. The strategy of midcourse correction, or periodic recomputation of nominal optimal controls, with some intermediate point in the state trajectory taken as the initial state, requires extremely fast sensing networks and computers. For systems having relatively short characteristic times, this approach is infeasible.

The implementation of some form of real-time feedback control would drastically mitigate these disadvantages. There are various ways of producing a feedback control which preserve some measure of optimality. For example:

1. The optimal trajectory in state space is precomputed, and whatever control is necessary to minimize the deviation of the actual state trajectory from the precomputed state trajectory is used. If the performance functional contains a term representing expenditure of control effort, this scheme may produce results which are far removed from optimality. If only deviations in the state trajectory are important, this method is quite attractive.
2. An optimally adaptive scheme may be used, in which the precomputed optimal control is augmented in real time using the deviation of the actual state from the nominal state trajectory. The algorithm which produces the control augmentation as a function of the state deviation would be so designed as to render the augmented state trajectory/control combination as optimal.
3. Suboptimal feedback control, as opposed to trajectory approximation or adaptive control, does not make use of a precomputed nominal trajectory. Instead, the criteria for optimality are somewhat relaxed, so that a feedback control which meets the revised criteria may be computed as a function of the state. The control schemes considered in this paper are all of this type.

It should be emphasized that the techniques presented here may be applied to either linear or nonlinear systems, having any type of performance functional. In principle, implementation would require but the smallest of process control computers.

Techniques Based on the Lyapunov Functional

Here we concentrate on the development of suboptimal control techniques which generate feedback controls by minimization of the time derivative of a Lyapunov functional. Algorithms of this sort have been used with some success on lumped-parameter systems (Lapidus and Luus, 1967; Koepcke and Lapidus, 1961; Schlossmacher and Lapidus, 1971). In this work we extend the general technique to distributed systems.

Consider the general distributed system with boundary conditions

$$u_t = F(x, t, u, u_x, u_{xx}, v)$$
$$u(x, t_0) = u_0(x), \text{etc.} \tag{1}$$

and the general performance functional

$$J = \int_{t_0}^{t_f} \phi(u, v, t)\,dt \tag{2}$$

in which the desired state u_D is taken to be zero

$$u_D(x) \equiv 0 \tag{3}$$

The system may be linear or nonlinear, parabolic or hyperbolic, boundary or volume controlled. The performance functional may be of any type.

We can define a Lyapunov functional for the above system

$$V(t) = \langle u(x,t),\ Qu(x,t)\rangle \tag{4}$$

in which the inner product $\langle\ ,\ \rangle$ is the spatial integral inner product in Hilbert space. Therefore, in long form notation

$$V(t) = \int_\Omega \int_\Omega u(x,t)Q(x,\xi)u(\xi,t)\,dx\,d\xi$$

as opposed to the lumped case, in which

$$V(t) = u^T Q u$$

The operator Q in Equation (4) is taken as being self-adjoint and positive definite. The Lyapunov functional given by (4) is a quadratic norm of the deviation of the state at any time $u(x, t)$ from the desired state $u_D \equiv 0$. Of course this Lyapunov functional is not unique since there are many possible choices for the operator Q. For the particular type of weighted quadratic norm of the state employed here, any Q which is self-adjoint and positive definite is an admissible choice. The choice of the particular Q which would provide the best suboptimal control is an entirely separate problem which we do not consider here. For an introduction to the Lyapunov functional on infinite-dimensional and finite-dimensional spaces see Berger and Lapidus (1968) and LaSalle and Lefschetz (1961).

Instead of choosing our control in the classically optimal manner, that is, to minimize J, we will seek to maximize the rate of approach of the system to the desired state. This is done by choosing the control from a bounded convex set to minimize the time derivative of $V(t)$. When $\dot{V}(t)$ is as negative as possible, the system is understood to be approaching the desired state as quickly as possible.

Note that this approach does not consider the form of the performance functional, and therefore does not guarantee an optimal control in the sense of minimization of the performance index. Systems with some types of performance functionals may be more amenable to Lyapunov suboptimal control than systems with certain other forms

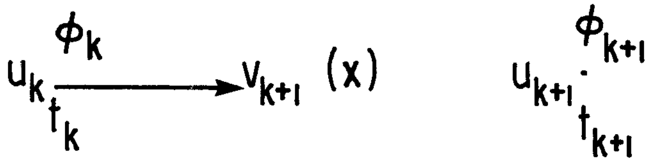

Fig. 1. Schematic representation of Lyapunov suboptimal control.

of the performance functional. For example, the Lyapunov functional approach seems the obvious choice for the minimum-time problem, while this approach would be clearly inadequate for a quadratic performance functional with greater relative weighting of the control term. When choosing a suboptimal control scheme, these factors should be kept in mind, as different schemes will be found more or less suitable for use on specific problems.

We now obtain the expression for $\dot{V}(t)$.

$$\dot{V}(t) = 2\langle u_t, Qu\rangle \tag{5}$$

$$\dot{V} = \Lambda = 2\langle F, Qu\rangle \tag{6}$$

$$\Lambda = 2\langle F(u, v, \dots), Qu\rangle \tag{7}$$

We now wish to make $\Lambda(t)$ as negative as is possible by choosing v, the control, from its admissible range. This is done by satisfying the equation

$$\nabla \Lambda_v = 0 \tag{8}$$

When F is linear in the control, the value for v satisfying (8) will be found to lie on the boundary of the admissible range, and a particularly simple form of the control law will result. When F is not linear in the control, Λ can be minimized without linearizing the system by using a direct search over the possible values of v.

Whether F is linear or nonlinear in the control, the basic procedure for generating the feedback control will be the same. This procedure is now presented.

1. Quantize the temporal domain of the system into a set of discrete sampling points t_k. A value for the state and a value for the time derivative of the Lyapunov functional will be associated with each of these points, as is schematically represented in Figure 1.

The value of Λ_k depends upon $v_k(x)$, the control chosen for the time interval $t \epsilon (t_k, t_{k+1}]$. The control is a constant with respect to time over each subinterval but may vary with x. The value of the control at any point in space and time is considered to lie between an upper and a lower limit

$$v_+ \leq v_k(x) \leq v^+$$

2. At each sampling point, the state is determined via a network of sensors and then $v_k(x)$ is chosen to minimize Λ_k. Since the control $v_k(x)$ depends upon the state $u_k(x)$ only, we have a true feedback control scheme. The feedback control is not optimal, because no account is taken of the actual performance index in the control problem. There is no provision for minimizing the amount of control effort expended in Lyapunov suboptimal control. Even if the performance index were the time integral of the Lyapunov functional

$$J = \int_{t_0}^{t_f} V(t)\,dt$$

minimization of $\dot{V}(t)$ for all t is not necessarily the same as minimization of J, since $V(t)$ as a function of $v(x, t)$ must follow a trajectory determined by the form of the system equation.

We shall now consider specific examples to illustrate the minimization of Λ_k by two methods, the linear gradient Lyapunov (LGL), and the direct search Lyapunov (DSL).

Consider the system

$$u_t = \nabla^2 u + f(u, v)$$

$$u(x, t_0) = u_0(x)$$

$$u(0, t) = u(1, t) = 0$$

and the admissible control range

$$v_+ \leq v \leq v^+$$

We define the Lyapunov functional as a spatial integral inner product

$$V = <u, Qu>$$

and form the time derivative

$$\Lambda = 2<\nabla^2 u, Qu> + 2<f(u, v), Qu> \quad (10)$$

Assuming that the system is linear in the control, we may write for each sampling point

$$\Lambda_k = 2<\nabla^2 u_k, Qu_k> + 2<f_v(u_k)v_k, Qu_k> \quad (11)$$

We now form the gradient of Λ_k with respect to v_k.

$$\nabla \Lambda_k = 2f_v(u_k)\ Qu_k \quad (12)$$

Note that $\nabla\ \Lambda_k$ is a function of x, and that (12) does not involve $v_k(x)$. Our control law becomes

$$v_k(x) = \begin{matrix} v^+ \\ 0 \\ v_+ \end{matrix} \quad \text{when} \quad f_v(u_k(x))\ Qu_k(x) \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \quad (13)$$

Therefore, $v_k(x)$ is a piecewise continuous function of x, composed of segments having one of the three values v_+, 0 or v^+. The control law (13) is analogous to the law which one obtains in the lumped case, except that v_k is a continuous function of one variable rather than an n-dimensional vector.

For the system

$$u_t + \alpha u_x = f(u, v)$$

$$u(x, t_0) = u_{ss}(x) \quad u(0, t) = u_0$$

which is a first-order linear hyperbolic system, the Lyapunov control law obtained is identical to (13). For the case where only one point in the spatial domain of the state is monitored, the following control law is obtained

$$v_k = \begin{matrix} v^+ \\ 0 \\ v_+ \end{matrix} \quad \text{when} \quad \beta f_v(\overline{u}_k)\overline{u}_k \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \quad (14)$$

In this control law, $\overline{u}_k$ is the scalar value of the state at a single point, and β is a constant greater than zero, which characterizes the Lyapunov functional for the single point measurement

$$V(t_k) = \beta\, \overline{u}_k^2 \quad (15)$$

Essentially (14) is a bang-bang control, dependent upon the state of the system at a single point. Later we will discuss multilevel bang-bang control for the first-order hyperbolic system.

For use with systems which are fully nonlinear, or at least nonlinear in the control, we have extended the method of Koepcke and Lapidus. Consider the general distributed system

$$u_t = F(x, t, u, u_x, u_{xx}, \ldots) \quad (16)$$

and its boundary conditions. The only thing that we require from this system in order to apply the method we are about to describe is that we have the ability to obtain $u(x, t)$ and that the control be a function of time only. To implement this direct search Lyapunov (DSL) method, we will consider

$$\Lambda_k = \dot{V}(t_k) \quad (17)$$

as opposed to

$$\Lambda_k = V(t_{k+1}) - V(t_k) \quad (18)$$

as considered by Koepcke and Lapidus. Since the time derivative of the state and thus of the Lyapunov functional, is an explicit function of the spatial derivatives of the state at any instant in time, we have no need to use the difference form of the time derivative given in (18). We form the Lyapunov functional

$$V = <u, Qu> \quad (4)$$

and the time derivative

$$\Lambda = 2<F, Qu> \quad (6)$$

then proceed to choose $v(t_k)$ from the admissible range

$$v(t_k) \in [v_+, v^+] \quad (19)$$

in the following manner:

(i) Determine $u_0(x)$, the initial state.

(ii) Select the value of v which yields the most negative Λ, using a simple search over the admissible range of v.

(iii) Apply the v resulting from step (ii) to the system for one time subinterval.

(iv) Repeat steps (ii) and (iii) until $t = t_f$ or until $V(t) < \epsilon$, a preselected tolerance.

Had we chosen (18) as the basis of the DSL method, we would have obtained

$$\Lambda_k = <(u_{k+1} - u_k), Q(u_{k+1} - u_k)> \quad (20)$$

We saw no advantage in going to this more complicated scheme, introducing the unknown u_{k+1} by approximation of Λ_k with its forward difference form.

Multilevel Bang-Bang Control of the First-Order Hyperbolic System

Here we describe a particularly simple suboptimal control scheme for first-order hyperbolic systems. Either linear or nonlinear first-order hyperbolic systems may be controlled using this method.

We note that the performance functional for *example B* is the time integral of the squared deviation of the outlet state from the desired value.

$$J = \int_{t_0}^{t_f} (u(1, t) - T)^2\, dt$$
$$u(1, t) = \overline{u} \quad (21)$$

The specific test problem we investigated, referred to as *example B*, is presented in Part II. The control, although applied over the entire spatial interval, is spatially independent. For the purposes of a suboptimal control algorithm, we restrict the control to lie between two limits.

$$v(t) \in [0, v^+],\ \forall\ \ t \quad (22)$$

In order to implement a two level bang-bang scheme, we define the following intermediate values of the control and state variables:

$$v^{\mp} \ni 0 < v^{\mp} < v^+ \quad (23a)$$

$$u^{\mp} \ni u_0 < u^{\mp} < T \quad (23b)$$

The control law is given as

$$v(t) = \begin{cases} v^{+} & \\ v^{\mp} & \text{when} \\ 0 & \end{cases} \begin{array}{l} u_0 \leq \overline{u} < u^{\mp} \\ u^{\mp} \leq \overline{u} < T \\ \overline{u} = T \end{array} \tag{24}$$

The character of the suboptimal control may be modified by adjusting the values of $u^{\mp}$ and $v^{\mp}$ within the ranges specified by (23). Also, this scheme may be generalized to as many levels as desired by defining sets of intermediate control and state values.

$$\{v_i^{\mp}\}_{i=1}^{n} \qquad 0 < v_i^{\mp} < v^{+}, \quad \forall \quad i \tag{25a}$$

$$\{u_i^{\mp}\}_{i=1}^{n} \qquad u_0 < u_i^{\mp} < T, \quad \forall \quad i \tag{25b}$$

and stating the control law as

$$v(t) = \begin{cases} v^{+} & \\ v_i^{\mp} & \text{when} \\ 0 & \end{cases} \begin{array}{l} u_0 \leq \overline{u} < u_n^{\mp} \\ u^{\mp}{}_{i+1} \leq \overline{u} < u^{\mp}{}_{i-1}, \\ \overline{u} = T \end{array} \qquad i = 2, \ldots, n-1 \tag{26}$$

To briefly describe the function of (24), the method provides maximum control effort when $\overline{u}$ is far away from T, the desired value, less control when $\overline{u}$ is in a certain neighborhood of T, and no control when $\overline{u} = T$. Numerical results obtained from this two level scheme will be presented in Part II.

Suboptimal Control of Parabolic Systems by Instantaneous Minimization of the Performance Functional Kernel

Consider a parabolic system which may or may not be linear, with a general performance functional

$$u_t = f(x, t, u, u_x, u_{xx}, v)$$

$$u(x, t_0) = u_0(x) \qquad + \text{ spatial b.c.}$$

$$J = \int_{t_0}^{t_f} \phi(u, v, t)\, dt$$

Instead of using an off-line control synthesis routine which produces an optimal control $v^0(x, t)$, we relax the criteria for optimality to such an extent that the computation of a feedback control becomes possible. The classically optimal control is one which minimizes

$$J = \int_{t_0}^{t_f} \phi(u, v, t)\, dt$$

The criteria for optimality of a control minimizing J may take the form of a two point boundary value problem relating the control, state, and adjoint variables over all space and time.

In formulating the revised criteria for suboptimal feedback control, we seek to satisfy two conditions:

1. The criteria to be met by the suboptimal control must approximate the criteria met by the optimal control to a reasonable extent.

2. The criteria for the suboptimal control must yield an expression for the control as a function of the state, at and/or before the instant in time under consideration. That is, our suboptimal control criteria should give rise to a feedback control law.

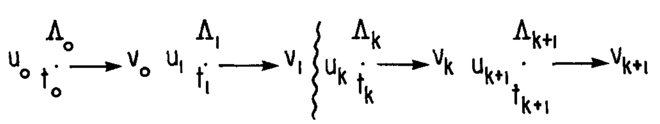

Fig. 2. Schematic representation of IKM suboptimal control.

Some criteria for suboptimal control which appear to satisfy the two conditions above are now presented. All involve the temporal kernel of the performance functional ϕ.

1. minimize $\phi(u, v, t)$; $\forall \ t$
2. minimize $\dot{\phi}(u, v, t)$; $\forall \ t$
3. ensure $\dot{\phi}(u, v, t) \leq 0$; $\forall \ t$
4. minimize $\phi(t) + \int_{t_0}^{t} \phi(t)dt$; $\forall \ t$
5. minimize $\dot{\phi}(t) + \phi(t)$; $\forall \ t$
6. minimize $\dot{\phi}(t) + \int_{t_0}^{t} \phi(t)dt$; $\forall \ t$

We chose to work with 1 because it is representative of the range of suboptimal control criteria and easily yields a suboptimal control law. In certain cases, however, one of the other criteria may provide a control law yielding superior performance. This question is open to investigation.

Now consider Figure 2.

Given a system at time t_k in state $u_k(x)$ and characterized by ϕ_k, we seek a control $v_{k+1}(x)$ to be applied to the system over the interval $t \in (t_k, t_{k+1}]$, which will transfer the system to the state u_{k+1} and which will yield a minimal value for ϕ_{k+1}. Briefly, then, our goal is to minimize $\phi_{k+1}(u_{k+1}, v_{k+1}, t_{k+1})$ for all k.

In order to select $v_{k+1}(x)$, we will use techniques of the classical calculus of variations, that is, we will set

$$\delta\phi_{k+1} \equiv 0 \tag{27}$$

We expand the performance functional kernel to linear terms

$$\delta\phi_{k+1} = \langle\phi_{uk+1}, \delta u_{k+1}(x)\rangle + \langle\phi_{vk+1}, \delta v_{k+1}(x)\rangle \tag{28}$$

Setting $\delta\phi$ to zero, we obtain

$$0 = \langle\phi_{uk+1}, \delta u_{k+1}(x)\rangle + \langle\phi_{vk+1}, \delta v_{k+1}(x)\rangle \tag{29}$$

It is obvious that $\delta u(x)$ and $\delta v(x)$ are not independent but are related by the perturbation form of the system equation. If the system equation, or its perturbation form, is linear in the state or variations thereof, we may express the relationship between δu and δv in terms of the casual Green's function of the parabolic system

$$\delta u_{k+1}(x) = \int_{t_k}^{t_{k+1}} \int_{\Omega} \delta v_{k+1}(x)\, g(x, t_{k+1}; \xi, \tau)\, d\xi d\tau \tag{30a}$$

where $g(x, t; \xi, \tau)$ is the casual Green's function, considered by Stakgold (1968) and many others. Adopting a shorthand operator notation, then

$$\delta u_{k+1} = G_{k+1}\, \delta v_{k+1} \tag{30b}$$

and (29) becomes

$$0 = \langle\phi_{uk+1}, G_{k+1}\, \delta v_{k+1}\rangle + \langle\phi_{vk+1}, \delta v_{k+1}\rangle \tag{31}$$

Introducing the adjoint Green's operator

$$\langle(G^*{}_{k+1}\phi_{uk+1} + \phi_{vk+1}), \delta v_{k+1}\rangle = 0 \tag{32}$$

Since $\delta v_{k+1}(x)$ is arbitrary, then in order for (27) to hold

$$G^*\phi_u + \phi_v = 0, \quad \forall \quad k \tag{33}$$

Note that the operator G^* will be time-dependent, as we have used it (in a perturbation situation), and furthermore it may not even exist when the state equation is nonlinear. Rather than following the mathematically rigorous (and computationally complex) drill of successive relinearization and calculation of an approximate numerical *Green's function* for each temporal subinterval, we chose to approximate the effect of G^* by a single, time-invariant multiplicative constant $1/\eta$. Thus, our approximate control law becomes

$$\eta \phi_{vk+1} + \phi_{uk+1} = 0 \tag{34}$$

We may take η to be any scalar, whose choice is left open. Equation (34) is a unique feedback control law, relating $v_{k+1}(x)$ to $u_{k+1}(x)$, with a single parameter η.

Consider the following brief example:

$$\tau = \int_{t_0}^{t_f} \langle u, Qu \rangle + \langle v, Rv \rangle \, dt$$

$$\phi(t) = \langle u, Qu \rangle + \langle v, Rv \rangle$$

Equation (34), the feedback control law, becomes

$$\eta R v_{k+1} + Q u_{k+1} = 0$$

or

$$v_{k+1} = -\frac{1}{\eta} R^{-1} Q u_{k+1}$$

If Q and R are completely continuous, a unique relation between v_{k+1} and u_{k+1} exists.

We have one more problem to solve before we have a workable suboptimal control scheme. Consider the case

$$u_t = u_{xx} + v \qquad + \text{spatial b.c.}$$

$$\phi = \mu \langle u, u \rangle + \gamma \langle v, v \rangle$$

The control law becomes

$$\eta v_{k+1}(x) = -\alpha u_{k+1}(x) \qquad \alpha = \mu/\gamma, \quad \eta > 0 \tag{35}$$

In (35), v_{k+1} is a function of u_{k+1}, which is not yet known when $t = t_k$. In a continuous time case, when the time subintervals are infinitesimally small, and $t_k \cong t_{k+1}$, (35) is nothing but a proportional control with negative feedback and gain α/η.

$$v(x) = -\frac{\alpha}{\eta} u(x); \quad \forall \quad t \tag{36}$$

Other suboptimal control criteria may reduce to other simple control modes in the continuous time case. Since we are working with discrete time, we must estimate $u_{k+1}(x)$ in order to calculate $v_{k+1}(x)$. We know that u_{k+1} is a function of v_{k+1}, that is,

$$F(u_k, v_{k+1}) = u_{k+1} \tag{37}$$

so we may write

$$v_{k+1}(x) = -\frac{\alpha}{\eta} F(u_k, v_{k+1}) \tag{38}$$

This is an implicit expression for v_{k+1} in terms of u_k, or a feedback control law for our specific case. We may obtain an estimate of u_{k+1} in various ways, for instance, the classical explicit difference net for a parabolic equation may be used to obtain the estimate $u_k(x_i)$. We rejected this approach out of hand, due to the large truncation error associated with the classical explicit method for parabolic PDE's (Saul'yev, 1964).

Instead, we chose a simple iterative method to find v_{k+1} based on the principle of contraction mapping (Kolmogorov and Fomin, 1957; Vermeychuk, 1972).

$$v^{j+1}_{k+1}(x) = -\frac{\alpha}{\eta} F(u_k(x), v^j{}_{k+1}(x)) \qquad j = 1, \ldots, N \tag{39}$$

The control scheme is carried out as follows:

(i) The state at time t_k, $u_k(x)$ is obtained via a sensing ensemble.

(ii) The classical explicit method for parabolic PDE's is used to obtain a starting value for $v_{k+1}(x)$.

$$v^1{}_{k+1}(x_i) = \frac{-\alpha/\eta\omega}{1 + \alpha(\Delta t)/\eta} [u_k(x_{i-1}) + u_k(x_{i+1}) + (\omega - 2)\, u_k(x_i)]; \quad \omega = (\Delta x)^2/\Delta t \tag{40}$$

(iii) Equation (39) is applied N times.

(iv) The resulting $v_{k+1}(x)$ is applied to the system as a feedback control.

The numerical method used to evaluate F in (39) is selected to suit the nature of the system under consideration. If the system is linear, as is *example* A, Saul'yev's average method will be quite sufficient. If the system is nonlinear, such as is *example* D of Part II, the method described by Vermeychuk (1972) or another special method will be required.

An interesting variation of the control law results when the state is a function of both space and time, and the control is a function of time only, as in *example* C of Part II, where

$$u_t = u_{xx} + \psi(x) v(t)$$

$$\phi = \mu \langle u, u \rangle + \gamma v^2$$

The control law used is

$$v_{k+1} = -\frac{\alpha}{\eta} \int_{-1}^{+1} u_{k+1}(\xi) \, d\xi \tag{41}$$

which reduces to a proportional feedback law based on the average value of the state over the spatial domain.

The two parameters η and N determine the performance of the instantaneous minimization method developed in this section. The parameter η must be adjusted as is the gain on a proportional controller. Some of the effects of variations in η and N will be examined in the discussion of numerical results. The next section will provide a brief comparison of the Lyapunov and instantaneous minimization methods.

Comparison of Lyapunov and Instantaneous Minimization Techniques

In this short section we present Table 1 and some comments comparing the Lyapunov and instantaneous minimization suboptimal control techniques. The multilevel bang-bang technique for the hyperbolic system is not included in the table because it is basically identical to multilevel bang-bang control in any lumped system, since only one spatial point in the domain of the state is considered.

For the purposes of comparison, the specific control laws given in the table will correspond to the following system

$$u_t = u_{xx} + f(u, v)$$

$$u(x, t_0) = u_0(x) \qquad + \text{spatial b.c.}$$

$$J = \int_{t_0}^{t_f} \phi(u, v, t) \, dt$$

$$\phi = \mu \langle u, u \rangle + \gamma \langle v, v \rangle$$

TABLE 1. COMPARISON OF DISTRIBUTED SUBOPTIMAL CONTROL TECHNIQUES

Method	Lyapunov		Instantaneous minimization criterion (1)
	LGL	*DSL*	
key function	$V(t) = <u,Qu>$		$\phi(u,v,t)$
objective	minimize $\dot{V}(t_k) = \Lambda_k, \ \forall\ k$		minimize $\phi(t), \ \forall\ t$
control law	$\nabla \Lambda_k(v_k) \equiv 0$		$\eta\phi_{vk+1} + \phi_{uk+1} = 0$
spec. c.l.	$f_v(u_k)Qu_k \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \Rightarrow v_k(x) = \begin{matrix} v^+ \\ 0 \\ v_+ \end{matrix}$		$v_{k+1}(x) = -\frac{\alpha}{\eta} F(u_x, v_{k+1})$
applicability:		v_k min Λ_k	
$v = v(t)$	yes	yes	yes
$v = v(x,t)$	yes	no	yes
hyperbolic	yes	yes	yes
parabolic	yes	yes	yes
linear state	yes	yes	yes
nonlinear state	yes	yes	yes
linear control	yes	yes	yes
nonlinear control	no	yes	yes
adjustable parameters	Q	Q	η, N

The table also contains information as to the range of applicability of each method.

All these methods share the advantage of producing a feedback control using very little computational effort. Of the three, instantaneous minimization has the widest range of applicability. The DSL method is restricted to systems in which the control is a function of time only, while the LGL method is restricted to systems linear in the control.

One definite advantage held by the instantaneous minimization method is the fact that the actual performance functional kernel is used, whereas in a Lyapunov scheme the Lyapunov functional may bear little relation to the actual performance functional, thus degrading the quality of the resulting suboptimal control.

Part II. Discussion of Test Systems and Numerical Results

The suboptimal control algorithms developed in Part I are tested on four example problems to obtain some indication of their performance. Results are presented and compared with results obtained from the application of classical open-loop optimal control synthesis algorithms to the same test problems.

The suboptimal feedback control algorithms are found to give acceptable performance on both linear and nonlinear systems at the expense of very little computational effort.

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SCOPE

The performance of the suboptimal control algorithms developed in Part I of this work is to be evaluated by direct comparison of the effect of suboptimal control to the effect of classically optimal control upon four specific test problems.

The basic test systems are a linear parabolic system, a linear first-order hyperbolic system, a linear parabolic system with variable coefficients, and a nonlinear parabolic system.

These four classes encompass a great number of the distributed systems encountered in engineering practice and are thus sufficient to provide the indications of performance we desire.

It should be noted that a rigorously optimal control for a linear system with a quadratic performance functional may be obtained in feedback form but only at the cost of extensive computation. No method for the computation of